Number Theory Multiplication
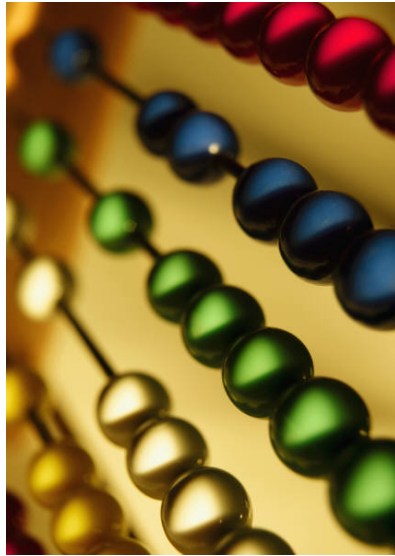
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Bemidji School District
Northern Elementary
Third Grade

Executive Summary

This unit is designed to be an introduction to multiplication. Students will be discussing equal groups, repeated addition, arrays, skip counting, and multiples by playing a variety of games and activities. Students will also have an introduction to the commutative property, the associative property, and the distributive property. After this unit students should have a good understanding of multiplication to support their work on the MCA Math Test. Example of MCA test question are located in the pretest and post-test. Some of the games and activities shared in this unit are from the series EVERYDAY MATH Grades K-6 by Wright Group/McGraw-Hill. This unit aligns with the Minnesota K-12 Academic Standards under the Number and Operation Strand for Third Grade. The standards specifically built into this unit are:

Number and Operation

> * Add and subtract multi-digit whole numbers; represent multiplication and division in various ways; solve real world and mathematical problems using arithmetic.
>
> > - 3.1.2.3 Represent multiplication facts by using a variety of approaches, such as repeated addition, equal sized groups, arrays, area models equal jumps on a number line and skip counting. Represent division facts by using a variety of approaches, such as repeated subtraction, equal sharing and forming equal groups. Recognize the relationship between multiplication and division.
> >
> > - 3.1.2.4 Solve real-world and mathematical problems involving multiplication and division, including both "how many in each group" and "how many groups" division problems
> >
> > - 3.1.2.5 Use strategies and algorithms based on knowledge of place value, equality and properties of addition and multiplication to multiply a two- or three digit number by a one digit number. Strategies may include mental strategies, partial products, the standard algorithm, and the commutative, associative, and distributive properties.

Table of Contents

Day 1 – Pretest

MN Standards 3.1.2.3, 3.1.2.4, and 3.1.2.5

Objective: Students will take the unit pretest so the teacher can assess their prior knowledge of the topics being covered.

Materials:
- A copy of the Pretest for each student
- Pencil

Launch: Tell the students they will be taking a pretest on multiplication strategies such as equal groups, repeated addition, and arrays. They will also be tested on the commutative property, associative property, and distributive property relating to multiplication. The teacher will encourage their students to do their best even though they may have not been taught that particular skill yet.

Explore: Allow students to work independently on their pretest. This test should not take the whole math period, but students will be allowed to finish at their pace.

Share: When the whole class completes the pretest, have the students get in groups of 2-4 and discuss what parts of the test they understood and what parts were difficult for them. Have one group member record their results.

Summarize: Have a class discussion about the pretest, and discuss difficulties. Let the students talk about parts of the test they struggled with and see if their classmates had any strategies to help with those problems. Tell them that we will be covering these topics over the next few weeks.

Day 2 – Dot Patterns

MN Standards: 3.1.2.3

Objective: To have students explore a variety of ways to represent the value of the dot pattern
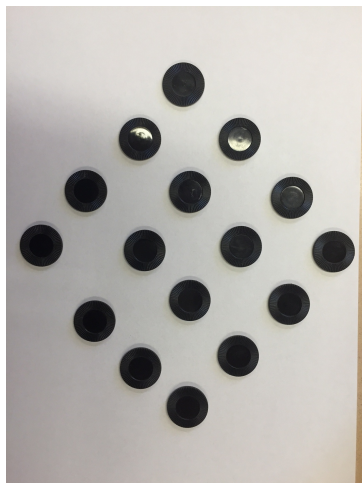
Materials:
- Variety of Number Dot Patterns
- Paper/pencil or Individual whiteboards

Launch: Display a dot pattern for the class to see. Then ask students how many dots they see? Students will use their whiteboards to show how they counted the dots on the pattern. Have students share their answers and how they counted the dots.

Explore: Have the students break into small groups and give each group a different dot pattern. As a group have students find as many different ways to show how they counted the number of dots on the dot pattern.

Share: The teacher will have each group share with the class the different ways they counted the dot patterns. The teacher should also have the students record their examples on the board. Then the teacher will encourage classmates to ask questions on what they found to be the quickest or the easiest.

Summarize: This is an introduction to multiplication. Students explored different ways to count dot patterns by using equal groups, repeated addition, and patterns. This will hopefully give students a better transition to multiplication.

Day 3 – Equal Groups with Blocks

MN Standards: 3.1.2.3

Objective: Students will be given a number of blocks and are asked to put them into equal groups.
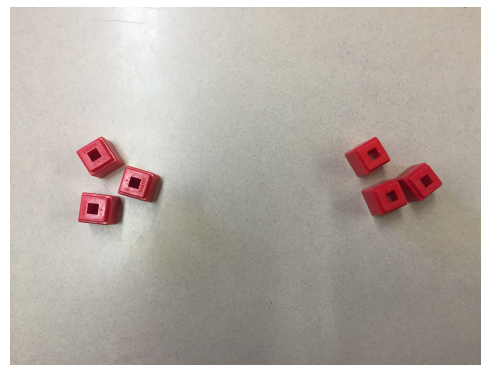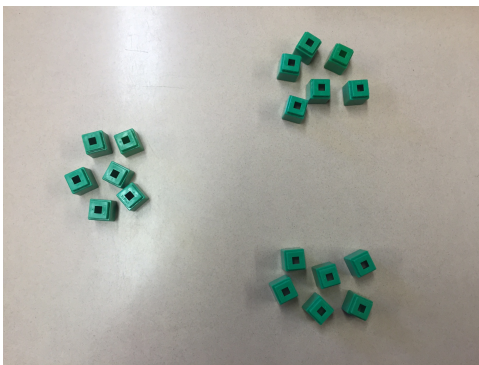
Materials:
- Pattern Blocks
- Whiteboard

Launch: Give each group of students a set of blocks. Have students explore with the blocks and find different variation of equal groups. Have students share their findings by recording them on the board with pictures and/or numbers. (For Example; 2x4=8, 5x3=15, 6x5=30)

Explore: Put students into small groups. Each group will be given a different number. Students will try to find all possible of equal groups for the product given to them. (For Example; 1x12=12, 2x6=12, 3x4=12) When they find an equal group for their number they will record their answer on a piece of paper. (For example; 4x3=12 students would write 4 groups of 12)

Share: Students will share their findings with the class by displaying their blocks on the classroom visualizer. Students will also record their equal group writing on the board.

Summarize: Students are starting to find the relationship between equal groups and multiplication. Some students may also start to see the commutative property by using the same factors but changing the order to get the same product. Students are also able to relate 3 groups of 4 to the equation 3x4=12
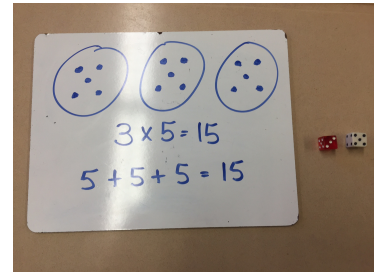
Day 4 – Equal Groups with dice

MN Standards: 3.1.2.3

Objective: Students will start to relate equal groups with multiplication by rolling a die to give them a number for how many equal groups and roll another die to tell how many in each group.

Materials:
- Dice (preferably one colored die and one white die)
- Whiteboard
- Paper/pencil to record who has a larger product

Launch: Students will be playing a multiplication game using equal groups. The student with the larger product will get a point. The student with the most points will win. To introduce this activity the teacher will play against a student on the large whiteboard in front of the class. The teacher would demonstrate first by taking two different colored dice. The teacher would roll them together and the colored die would be my number of groups and my white die tells me how many are in each group. For Example, if my colored die is 3 and my white die is five. The teacher would draw 3 circles for the equal groups and put in five dots in each circle. When the teacher counts all the dots he or she will find the product of 3x5=15. Students will write their equation below their proof drawing. This would also be a great time to introduce the words factor and product so students can start to use this terminology when talking with classmates. The teacher then would then let the student do the same process as my partner. We will then compare our drawings and equations to see who has the larger product.

Explore: The students will work in partnerships or small groups. They will record their equations in a table and circle the larger product. During this time the teacher should encourage students to read their multiplication equations to their classmates and discuss if it's better to have larger number of equal groups or a larger number in each equal group. A variation to this game is to have students write the repeated addition equation rather then the multiplication equation or both.

Share: When we come back to a group, the teacher will ask students to share if it made a difference if they had a larger number of groups or a larger number in each group. The teacher should also encourage students to share any other patterns they observed while playing this game.

Summarize: Students will start to have a better understanding of how multiplication equations relate to equal groups. The first factor shows them how many groups (circles) and the second factor tells them how many are in each group (dots inside circles). Students can also show a repeated addition that matches the proof drawing as well as the multiplication equation.

Day 5 – Skip Counting/Multiples – 100 Chart

MN Standards: 3.1.2.3

Objective: Using a 100 chart to color-code the different multiples so that students can visually understand the patterns and relationship to multiplication.
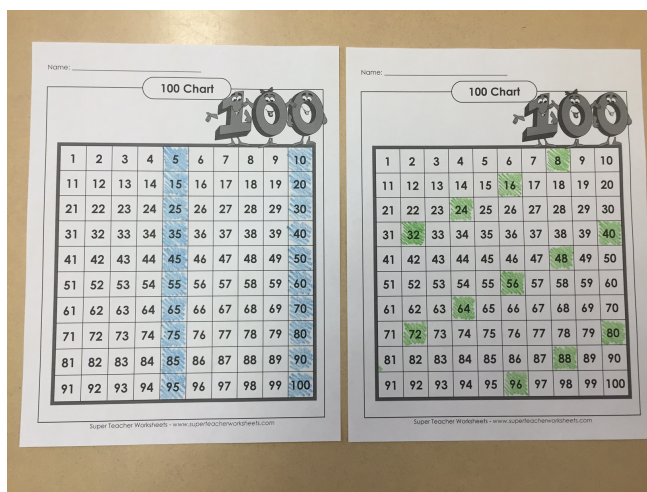
Materials:
- 100 Charts
- Crayons

Launch: Students will each get a 100 chart, Students will skip count together starting with 5s. When students are skip counting they should point to the number they are saying on there 100s chart. (For example; students would say 5, 10, 15 and point to them on their chart) Continue this process with the 10s, 2s, 4s, 6s, 8s, 3s, 9s, and 7s. Once students understand skip counting on a 100 chart.

Explore: Students will each be assigned to a number from 2 to 10. When they get a number they will skip count and color in each number they say. They will do this all they way through the 100s chart. Then group students where they each have a different number colored in. Students will discuss some of the similarities and differences in their 100s chart. Also see if they can find any more patterns that have not been discussed.

Share: Each group will be given a chance to show what their 100s chart looks like. Then share with the class some of the similarities and differences they have observed. Students will also share any patterns they may find. The teacher will also have students compare 100 charts that had the same multiple to see if there are any disagreements that need to be clarified.

Summarize: Students will be able to visually understand multiples of 2 through 10. Students will also connect skip counting to multiplication. Students will also visually see that some numbers will have more then one multiple.

Day 6 – Multiples/BUZZ Game

MN Standards: 3.1.2.3

Objective: To identify multiples of a number or numbers that contains a certain digit

Materials: None

Launch: This game can be used in groups of 5-10 or a whole group. Students will form a circle within their group. A leader will be assigned. This student names a whole number from 3 to 9. This number is the BUZZ number. The leader also chooses the STOP number. The STOP number should be at least 30. The player to the left of the leader begins the game by saying "1". Play continues clockwise with each player saying either the next whole number or BUZZ. A player must say BUZZ instead of the next number if: The number is the BUZZ number or a multiple of the BUZZ number; or the number contains the BUZZ number as one of it's digits. If a player makes an error, the next player starts over with "1". Play continues until the STOP number is reached. For the next round, the player to the right of the leader becomes the new leader.

Explore: Example of this process is: The BUZZ number is 4. Play should proceed as follows: "1, 2, 3, BUZZ, 5, 6, 7, BUZZ, 9, 10, 11, BUZZ, 13, 14, 15, BUZZ and so on.

Share: Students will share challenges they had while playing this game. Students will share what patterns or techniques they learned that made it easier to the play the game. Students will also tell what patterns they noticed when working with multiples.

Summarize: This activity is to help students understand multiples of numbers 3 through 9. This is also a challenge to understand multiplies without having a visual aid to help them see the patterns. They have to rely on each other's communication as well as their understanding of multiplication.

Variation of the BUZZ game called BIZZ – BUZZ
Bizz-Buzz is played like Buzz, except that the leader names 2 numbers: a BUZZ number and a BIZZ number. Players say BUZZ instead of the number if the number is a multiple of the BUZZ number. Players will say BIZZ instead of the number if they number is a multiple of the BIZZ number. Players will say BIZZ-BUZZ instead of the number if the number is a multiple of both the BUZZ number and the BIZZ number. (For Example, The BUZZ number is 6, and the BIZZ number is 3. Play should proceed as follows: "1, 2, BIZZ, 4, 5, BIZZ-BUZZ, 7, 8, BIZZ, 10, 11, BIZZ-BUZZ, 13, 14, BIZZ, 16" and so on.)

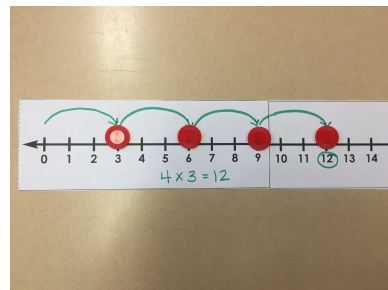This game is from EVERYDAY MATH Games Grades K-6

Day 7 – Number line – Equal Jumps

MN Standards: 3.1.2.3

Objective: Students will use a number line to show another way to show equal groups also known as equal jumps.

Materials:
- A number line 1-100
- Small counters
- Whiteboards

Launch: Each student needs to have a number line and a small handful of counters. The teacher will relate back to skip counting in the 100s chart and say we are going to do the same with a number line. So starting with 5s the teacher should start counting and point to the multiples of 5 as he or she works their way down the number line. Then the teacher will do this again with a few more multiplies so that all students have a clear connection and understanding of number lines and skip counting. Then the teacher will give students a multiplication equation. (For example 2x3) Students would refer back to our equal groups activities that the factor 2 represents how many groups and the factor 3 represents how many are in each group. The students will do the same with the number line. The factor 2 tells me how many counters to use or how many equal jumps I will make. The factor 3 tells me how numbers to jump for each group. So starting on my number line the teacher will have two counters in her or his hand. The first counter will go on 3 the second counter will go on 6. The teacher has no more counters so that means my product of 2x3=6. You can also connect these equal jumps by putting arrows above the digits showing you are jumping from 0 to 3 and 3 to 6.

Explore: Students will continue this process with a partner. One partner will roll two dice and use one die for each factor. This partner will also write the multiplication equation on a whiteboard. The second partner will show their work on the number line using counters and/or arrows to show the equal jumps.

Share: The class will come together and discuss how equal jumps relate to other activities we have done in the past. Also share ideas on how to make this activity more challenging or quicker for students to share their work.

Summarize: Students will have a chance to review equal groups by learning a new method. This activity also is a close relation to the 100s chart. Students will visually understand that a number line is just a 100s chart put all on one line. Number lines are also a familiar tool that students are use to using to work on addition and subtraction so will hopefully be helpful in understanding multiplication.

Day 8 – Arrays using Legos

MN Standards: 3.1.2.3 and 3.1.2.5

Objective: Students will use Legos to introduce multiplication arrays
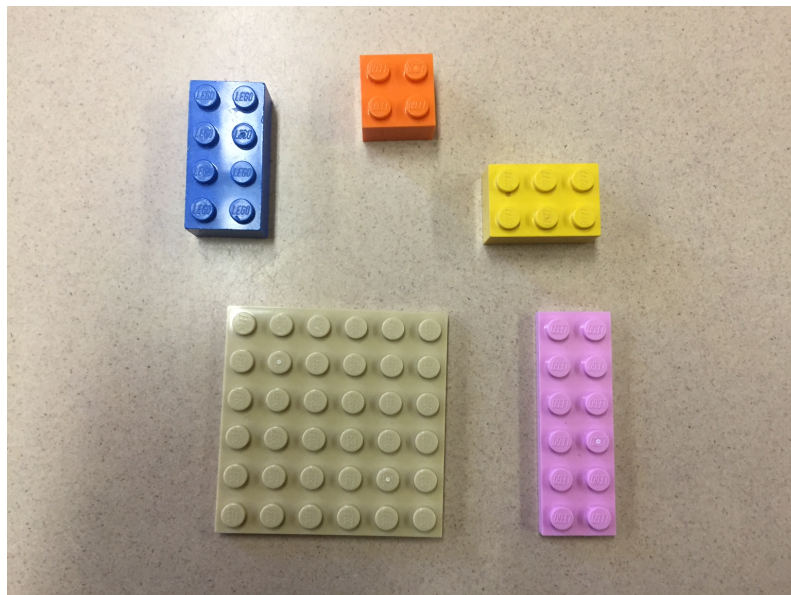
Materials:
- A variety of Lego pieces
- Whiteboard

Launch: Give each group of students a bucket of Legos. Encourage the students to pull out the Legos and play with them. While they are playing with them have students share what they observe about the Legos. Encourage the students to share that they see equal groups, rows, and columns. Introduce to the students that organized rows and columns of equal groups of dots are called an array.

Explore: Have students explore their bucket of Legos and see how many different kinds of arrays they can find. As a group students will keep a list and a drawing of all the arrays they have found.

Share: Have each group share the different arrays they found. Some students may be able to show the commutative property by saying that 2x5 is a different array as 5x2 but they have the same product. Students will also relate back to past projects and notice skip counting, multiplies, equal groups and repeated addition.

Summarize: This activity allows students to use a real life toy that most children own or have had the chance to play with in their lifetime. They can use this familiar object to help them understand multiplication in many different ways. Legos are also very organized and concrete objects that show rows and columns such as arrays.

11

Day 9 – Arrays with cm graph paper

MN Standards: 3.1.2.3 and 3.1.2.5

Objective: Students will draw varies arrays using cm grid paper.
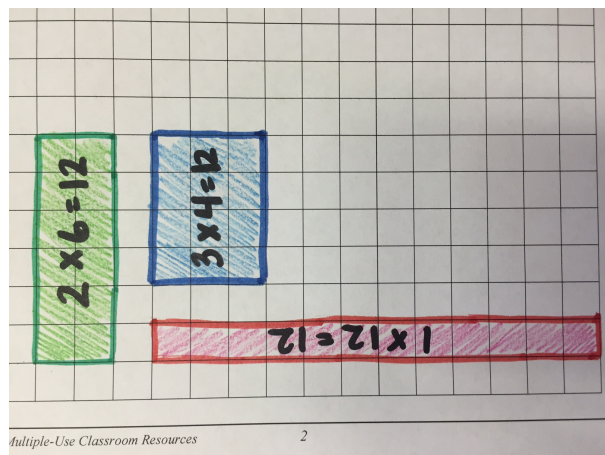
Materials:
- cm grid paper
- Cards or dice
- Crayons/colored pencils
- Pencil

Launch: The teacher will begin by reviewing arrays using Legos from the previous activity.   Then the teacher will display a Lego piece and have students draw the same area onto their cm grid paper. The teacher will encourage students to share how they drew the array. This will lead the class into the discussion that each side of a cm grid paper is considered 1 unit. This understanding will make sure that all drawings will be drawn to the same scale. To help students who are struggling have them put dots in each box like the array they see on the Lego. Then they can trace the lines around the dots to give them the array shape.

Explore: Once again allow the students to explore the bucket of Legos and see how many different array drawings they can make on their cm grid paper. Have students label each of their arrays with the correct multiplication equation.

Share: Have a variety of students share how they drew and colored their arrays in. As a class we will discuss some of the patterns, similarities, and differences observed. Also have students share ways on how they organized their answers and how they thought out the problem in their head.

Summarize: Using cm grid paper is another useful tool to help examine multiplication. This is a more 2D method that helps students draw proof drawings to support their answers.

Day 10 – Array City

MN Standards: 3.1.2.3 and 3.1.2.5

Objective: This activity is to show what students have learned about arrays and relate them to real life objects.

Materials:
- Large poster paper
- construction paper (variety of colors)
- Scissors
- 1 in squares precut (variety of squares)
- Glue stick
- Pencil/crayons

Launch: Have students think about large cities they have visited in person, read about, or have seen on Television. Discuss the characteristics of different buildings they have seen. Some students may say tall, short, wide or skinny. Also encourage students to express their observation of the windows located on these city buildings. Relate these windows to arrays that we have been working on for the past couple of days. Students are going to use what they know and have learned about multiplication and make a city out of different arrays. Every group is going to use 1 in squares to keep size scales of the cities similar.

Explore: Students will use 1 in squares to make and array. They will then take that array and make it into a building. Once they finish a building they will start to join these array building to make an array city.

Share: Allow each group of students to share their array city. Have students explain the way they organized their buildings in the city. Have students explain how they kept track of which arrays they hadn't made yet. Did they have array buildings that were 2x4 and 4x2? Do these two buildings look the same? Do they get the same product?

Summarize: This activity allows students to build their multiplication fact fluency. They also learned to organize their data to help them make as many arrays as possible without duplicating. Students also were able to relate arrays to a real life objects.

Day 11 Commutative Property – High Roller

MN Standards: 3.1.2.3 and 3.1.2.5

Objective: To roll the highest number

Materials:
- 2 dice
- whiteboard or paper/pencil
- Multiplication table

Launch: Students will work in groups of 2 to 4 students. One player rolls the dice. The player keeps the die with the larger number (the High Roller) and throws the other die again. The player then multiplies the number from the first die to the number on the second die to get the product of the two dice. Each player repeats these steps. The winner is the player with the highest number after two rounds

Explore: Students will work in small groups and play the game High Roller. During this time the teacher will monitor groups- listening for vocabulary that directly relates to multiplication such as equal groups, multiples, and repeated addition.

Share: Students share their results of their games. Discuss with students the highest product they got. Is this the highest product they can get? What was their lowest product possible? Can you get a lower product then what you found? Did anyone get the same product but different factors? Did anyone get same product with the same factors but in a different order?

Summarize: After exploring a variety of multiplication equations student will start to observe that because of the commutative property that 3x4=4x3. Many students relate this to the turn around rule. If you just switch the factors around you will get the same product. For a final visual on commutative property the teacher would display a multiplication table and discuss with students the patterns between the top diagonal of the multiplication table and the bottom diagonal.

Day 12 – Area Models

MN Standards: 3.1.2.3 and 3.1.2.5

Objective: Students will play an area model game where they want to cover as much of the cm grid paper with their color as possible.
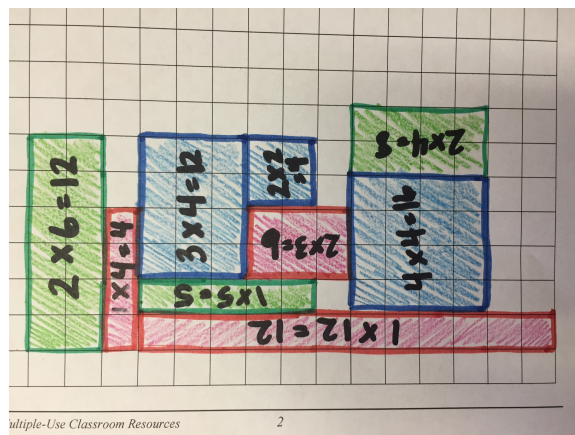
Materials:
- cm grid paper
- A different color crayon for each student in a group
- 2 dice

Launch: Students will each choose a color. Students will then roll two dice. Using the numbers found on the dice students will make an array that matches the numbers on the two dice. Students will relate area models to the array models we did on the cm grid paper. Students are also allowed to use the commutative property to change their area model if needed. If your dice come up with an array that does not fit anywhere on your cm grid paper without overlapping another area model, that player loses their turn. When no more area models can be made the player with the most area colored in on the cm grid paper wins the game.

Explore: Have students work in pairs to begin. Then allow student to join up with another group or make groups of three to see if student's strategies change or if it's harder or easier to control the game when more players are involved. Go around to groups and see if any students have a strategy to win the game.

Share: Have different groups share their area model drawings. Did any groups find anything more difficult then others? Did anyone find a way that will allow him or her to win every time? What numbers gave you the biggest area? What numbers gave you the smallest area?

Summarize: Students will be able to identify multiplication with area. This activity also gives you a connection to the area of a rug you are trying to cover up. This helps students connect area to real life situations. Area models are also a great way to show the commutative property.

Day 13 – Product Comparing

MN Standards: 3.1.2.3 and 3.1.2.5

Objective: Students to figure out the products they don't know by using facts that they do know.

Materials:
- 2 sets of cards only using numbers 0-9
- 36 counters for making groups

Launch: Each player gets a set of 0-9 cards. Separate your cards into one stack of 0-4cards and another stack of 5-9 cards. Place each stack face down on the table. Each player turn over one card from each stack and multiplies the two numbers. Use mental math or counters to find the product. (State the multiplication fact, 3x7=21 or 7x3=21) The player with the greater product takes all four cards. If there is a tie, each player turn over two more cards and multiplies those two numbers to see who takes all the cards. After all the cards are drawn, the player with the most cards is the winner.

More games:
A. Use 4 sets of 0-5 cards in a face down stack. Draw two cards at a time and compare the products.
B. Use 4 sets of 4-9 cards in a facedown stack. Draw 2 cards at a time, compute and compare the products.

Explore: Students will work with partners with the activity listed above. Encourage students to try and use different strategies to solve the equation before grabbing the counters. Have students think out loud to their partner so they are able to learn or help each other through the process.

Share: Students will share other strategies they learned that helped them solve the multiplication equation using mental math. (For Example: what is the product 4x6=, students could say 2 sixes equal 12 so 4 times 6 is 24 or I know 4x5=20 and I had 4 more ones to make it 24.)

Summarize: Students are starting to find their own ways to help them solve multiplication equations. This is also a great lead into activity for the distributive property.

This game is from EVERYDAY MATH Games Grades K-6

Day 14 – Distributive Property with Legos

MN Standards: 3.1.2.3 and 3.1.2.5

Objective: Using Legos students will be able to identify the distributive property by breaking down one large array into a few smaller arrays.
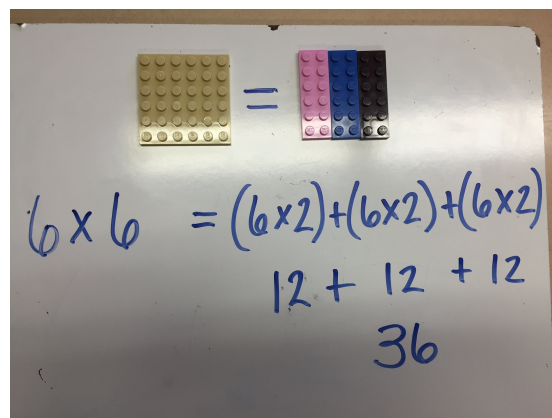
Materials:
- Variety of Legos
- Cm grid paper
- Paper/pencil
- Crayons

Launch: Students will be given a Lego with a large product, like 6x4. This can be discouraging and frustrating. Ask students if they can make that same array with using more then one Lego. Ask a student to display their new array and compare it to the original area. This student chose to use (6x2)+(6x2)=6x4. Another student made a new array (2x4)+(2+4)+(2+4)=6x4. All three of these arrays come to a product of 24. They are all correct just the students who made more arrays within the large array worked with smaller numbers, which made it solving the equation easier.

Explore: Display a new area for students to explore with their Legos. See if students can build a new array using more arrays but has the same shape and product as the original. Have student record and share their answers with the class after each problem. Slowly work your way up to larger beginning arrays such as a 2-digit number multiplied by a one-digit number.

Share: Have students share what they would do if one side of the array were a double-digit number? Would anything change? Can students use the distributive number make solving multiplication problems easier or tougher, and why? Does the distributive property take more time or less time to solve the answer? Does anyone have a way to model this on paper?

Summarize: Students are given a clear introduction model of the distributive property using concrete manipulative such as Legos. Students have also already built confidence in working with Legos from past activities. Once again Legos are a great visual tool to help students connect to multiplication.

Day 15 – Number Stories

MN Standards: 3.1.2.3, 3.1.2.4, 3.2.2.1

Objective: Students will use their knowledge of multiplication and division to solve real-world mathematical problems using strategies taught in this unit.

Materials:
- Whiteboard or paper/pencil

Launch: Students will be asked the problem "If one quart of ice cream will serve eight people, how many people will five quarts serve?" Encourage students to draw a proof drawing for their solution as well as the equation. Allow students to share their different solutions on the board for the classmates to see. Repeat this process until students are comfortable with the task.

Explore: Students will work in small groups and work on 8 task cards. Students will record their proof drawings and equations on a paper divided into 8 rectangles. Students will be encouraged to use as many different proof drawings they can for each problem. Students may use counters if necessary, but must transfer their counters into a proof drawing.

Share: Have each group share their 8 rectangles with their proof drawings. Have them explain which proof drawing they used first and why they thought it was good for that number story.

Summarize: This activity allowed students to use the strategies they have learned in this unit and apply them to real world number stories.

Day 16 – Name that Number

MN Standards: 3.1.2.3, 3.1.2.5, 3.1.2.1, and 3.1.2.2

Objective: To collect the most cards by using addition, subtraction, multiplication and division strategies learned in this unit and as well as students prior knowledge.

Materials:
- Paper/pencil for each student
- Set of cards 0-9 (4 of each) and 11-20 (1 of each) total of 50 cards

Launch: Students will work in small groups, preferably groups of 2 to 3 students. One player shuffles the cards and deals 5 cards to each player. The dealer places the remaining cards number-side-down on the table, turns over the top card, and places it beside he deck.   This is the target number for the round. Players try to match the target number by adding, subtracting, multiplying, and dividing the numbers on as many of their cards as possible. A Card may be used only once. Players record their solutions on a sheet of paper. When players have written their best solutions and shared with their group. If a students solution is proven wrong that student does not get to set aside any of those cards. When all solutions in a group have been shared students will take turns doing the following: 1) Set aside the cards they used to name the target number. 2) Replace them by drawing new cards from the top of the deck. 3) Put the old target number on the bottom of the deck. 4) Turn over a new target number, and play another hand. Play continues until there are not enough cards left to replace all of the players' cards.   The player who sets aside more cards wins the game.

Explore: Students will be given time to play the game Name that Number. Students will be sharing their solutions with their classmates before moving on to the next round. Encourage students explore different strategies to help them win.

Share: Have each group share different things they learned while playing Name that Number. Did they catch anyone with an incorrect solution? Was it easier or harder to use 2 cards, 3 cards, 4, cards or all 5 cards? Did anyone use all 4 different operations in one solution? Have students share their equations on the board for classmates to see.

Summarize: Students are practicing basic addition, subtraction, multiplication, and division facts during this game. Students are also using their prior knowledge of addition and subtraction and are connecting it to what they have learned about multiplication and division.

This game is from EVERYDAY MATH Games Grades K-6

Day 17 — Post Test

MN Standards: 3.1.2.3, 3.1.2.4, and 3.1.2.5

Objective: Assess the students' progress made during the unit on multiplication.

Materials:
- A copy of the Post Test for every student
- Pencil

Launch: Tell the students they will be taking a posttest that is just like the pretest they took a couple of weeks ago. It will measure how much they have learned about the topics we've been covering.

Explore: Allow the students to work independently on the posttest. The posttest should take the majority of the hour.

Share: When everyone has completed the test, let the students share with a partner any problems that may have arisen on the test and let them discuss solutions.

Summarize: If any problems were troublesome for a majority of the class, discuss the problem and find out any common errors.

Multiplication Test

Name: _____

**Directions:** Read and solve each question. Show your work!

1. Draw a proof drawing that shows 3 groups of 5.

2. Show what 6x8 looks like using repeated addition.

3. Draw an array for the equation 4x4.

4. Draw your own number line starting at 0 and ending on 10. Then show me equal jumps for the equation 2x3.

5. What number is a multiple of 2 and 5?

6. Draw a proof drawing that has an area of 12 square units. (Hint: There is more then one correct answer)

7. Complete the multiplication sentence.

$$7 \times 3 = 3 \times \underline{\quad}$$

8. Find the area of the large rectangle by adding the areas of the two small rectangles. Show your work.
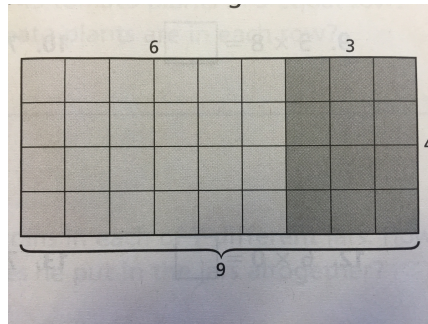
9. Using the numbers 7 5 8 2 10 only once, show me 2 different ways that you can make the target number 16. (You can add, subtract, multiply, divide as many times you wish)

10. Nina has four balloons. Her sister has nine times as many balloons. How many balloons does her sister have? Show your work with a proof drawing.